

# BIPOLYNOMIAL HILBERT FUNCTIONS

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**ABSTRACT.** Let  $X \subset \mathbb{P}^n$  be a closed subscheme and let  $HF(X, \cdot)$  and  $hp(X, \cdot)$  denote, respectively, the Hilbert function and the Hilbert polynomial of  $X$ . We say that  $X$  has *bipolynomial Hilbert function* if  $HF(X, d) = \min \{hp(\mathbb{P}^n, d), hp(X, d)\}$  for every  $d \in \mathbb{N}$ . We show that if  $X$  consists of a plane and generic lines, then  $X$  has bipolynomial Hilbert function. We also conjecture that generic configurations of non-intersecting linear spaces have bipolynomial Hilbert function.

## 1. INTRODUCTION

The Hilbert function of a scheme  $X \subset \mathbb{P}^n$  encodes a great deal of interesting information about the geometry of  $X$  and so the study of  $HF(X, \cdot)$  has generated an enormous amount of research. One of the most crucial and basic facts about the Hilbert function of a scheme is that the function is eventually polynomial. More precisely

$$HF(X, d) = hp(X, d), \text{ for } d \gg 0.$$

In general, knowledge of the Hilbert polynomial does not determine the Hilbert function. But, there are some interesting situations when this is the case. E.g. if  $X$  is a generic set of  $s$  points in  $\mathbb{P}^n$ , it is well known, and not hard to prove, that

$$HF(X, d) = \min \left\{ hp(\mathbb{P}^n, d) = \binom{n+d}{d}, hp(X, d) = s \right\},$$

for all  $d \in \mathbb{N}$ . A much harder result is due to Hartshorne and Hirschowitz. In [HH82] the authors considered schemes  $X \subset \mathbb{P}^n$  consisting of  $s$  generic lines and they proved that

$$HF(X, d) = \min \left\{ hp(\mathbb{P}^n, d) = \binom{n+d}{d}, hp(X, d) = s(d+1) \right\},$$

for all  $d \in \mathbb{N}$ .

Inspired by these results about points and lines, we restrict our attention to that special family of schemes known as *configurations of linear spaces*. We recall that a configuration of linear spaces  $\Lambda \subset \mathbb{P}^n$

is nothing more than a finite collection of linear subspaces of  $\mathbb{P}^n$ ; see [CCG09, CC09] and [DS02] for more on these schemes and their connection with *subspace arrangements*. We further say that a configuration of linear spaces is *generic* when its components are generically chosen.

The Hilbert polynomial of a generic configuration of linear spaces is known, thanks to a result of Derksen, see [Der07]. Thus, in light of the results on the Hilbert function of generic points and generic lines, we propose the following

**Conjecture:** *if  $\Lambda \subset \mathbb{P}^n$  is a generic configuration of linear spaces with non-intersecting components, then*

$$HF(X, d) = \min \{hp(\mathbb{P}^n, d), hp(X, d)\},$$

*for all  $d \in \mathbb{N}$ .*

We will call a Hilbert function defined as above *bipolynomial*. Hence, the conjecture states that generic configurations of linear spaces with non-intersecting components have bipolynomial Hilbert function.

As we mentioned above, this conjecture is true when  $\dim \Lambda = 0$  (generic points) and when  $\dim \Lambda = 1$ . The conjecture holds in the dimension one case because of the result about generic lines in [HH82] and because we know how adding generic points to a scheme changes its Hilbert function, see [GMR83].

In this paper we produce new evidence supporting our conjecture. Namely, we show that the union of one plane and  $s$  generic lines has bipolynomial Hilbert function.

The paper is structured as follows: in Section 2 we introduce some basic notation and results we will use; Sections 3 and 4 contain the base cases for our inductive approach; Section 5 contains our main result, Theorem 5.1. These sections are followed by a section on Applications and another in which we propose a possibility for the Hilbert function of any generic configuration of linear spaces, even one in which there are forced intersections.

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## 2. BASIC FACTS AND NOTATION

We will always work over an algebraically closed field  $k$  of characteristic zero. Let  $R = k[x_0, \dots, x_n]$  be the coordinate ring of  $\mathbb{P}^n$ , and

denote by  $I_X$  the ideal of a scheme  $X \subset \mathbb{P}^n$ . The Hilbert function of  $X$  is then  $HF(X, d) = \dim(R/I_X)_d$ .

**Definition 2.1.** Let  $X$  be a subscheme of  $\mathbb{P}^n$ . We say that  $X$  has a *bipolynomial Hilbert function* if

$$HF(X, d) = \min \{hp(\mathbb{P}^n, d), hp(X, d)\},$$

for all  $d \in \mathbb{N}$ .

It will often be convenient to use ideal notation rather than Hilbert function notation, i.e. we will often describe  $\dim I_X$  rather than  $HF(X, d)$ . It is clearly trivial to pass from one piece of information to the other.

The following lemma gives a criterion for adding to a scheme,  $X \subseteq \mathbb{P}^n$ , a set of reduced points lying on a linear space  $\Pi \subseteq \mathbb{P}^n$  and imposing independent conditions to forms of a given degree in the ideal of  $X$ .

**Lemma 2.2.** *Let  $d \in \mathbb{N}$ . Let  $X \subseteq \mathbb{P}^n$  be a scheme, and let  $P_1, \dots, P_s$  be generic distinct points on a linear space  $\Pi \subseteq \mathbb{P}^n$ .*

*If  $\dim(I_X)_d = s$  and  $\dim(I_{X+\Pi})_d = 0$ , then  $\dim(I_{X+P_1+\dots+P_s})_d = 0$ .*

*Proof.* By induction on  $s$ . Obvious for  $s = 1$ . Let  $s > 1$  and let  $X' = X + P_s$ . Obviously  $\dim(I_{X'+\Pi})_d = 0$ . Since  $\dim(I_{X+\Pi})_d = 0$  and  $P_s$  is a generic point in  $\Pi$ , then  $\dim(I_{X'})_d = s - 1$ . Hence, by the inductive hypothesis, we get  $\dim(I_{X'+P_1+\dots+P_{s-1}})_d = \dim(I_{X+P_1+\dots+P_s})_d = 0$ .  $\square$

Since we will make use of Castelnuovo's inequality several times in the next sections, we recall it here in a form more suited to our use (for notation and proof we refer to [AH95], Section 2).

**Definition 2.3.** If  $X, Y$  are closed subschemes of  $\mathbb{P}^n$ , we denote by  $Res_Y X$  the scheme defined by the ideal  $(I_X : I_Y)$  and we call it the *residual scheme* of  $X$  with respect to  $Y$ , while the scheme  $Tr_Y X \subset Y$  is the schematic intersection  $X \cap Y$ , called the *trace* of  $X$  on  $Y$ .

**Lemma 2.4. (Castelnuovo's inequality):** *Let  $d, \delta \in \mathbb{N}$ ,  $d \geq \delta$ , let  $Y \subseteq \mathbb{P}^n$  be a smooth hypersurface of degree  $\delta$ , and let  $X \subseteq \mathbb{P}^n$  be a scheme. Then*

$$\dim(I_{X, \mathbb{P}^n})_d \leq \dim(I_{Res_Y X, \mathbb{P}^n})_{d-\delta} + \dim(I_{Tr_Y X, Y})_d.$$

$\square$

Even though we will only use the following lemma in the cases  $m = 2$ ,  $m = 3$  (see the notation in the lemma), it seemed appropriate to give the more general argument since such easily understood (and non trivial) degenerations occur infrequently.

**Lemma 2.5.** *Let  $X_1 \subset \mathbb{P}^n$  be the disconnected subscheme consisting of a line  $L_1$  and a linear space  $\Pi \simeq \mathbb{P}^m$  (so the linear span of  $X_1$  is  $\langle X_1 \rangle \simeq \mathbb{P}^{m+2}$ ). Then there exists a flat family of subschemes*

$$X_\lambda \subset \langle X_1 \rangle \quad (\lambda \in k)$$

whose special fibre  $X_0$  is the union of

- the linear space  $\Pi$ ,
- a line  $L$  which intersects  $\Pi$  in a point  $P$ ,
- the scheme  $2P|_{\langle X_1 \rangle}$ , that is, the schematic intersection of the double point  $2P$  of  $\mathbb{P}^n$  and  $\langle X_1 \rangle$ .

Moreover, if  $H \simeq \mathbb{P}^{m+1}$  is the linear span of  $L$  and  $\Pi$ , then  $\text{Res}_H(X_0)$  is given by the (simple) point  $P$ .

*Proof.* We may assume that the ideal of the line  $L_1$  is

$$(x_1, \dots, x_m, x_{m+1} - x_0, x_{m+3}, \dots, x_n)$$

and the ideal of  $\Pi$  is  $(x_{m+1}, \dots, x_n)$ , so the ideal of  $X_1$  is

$$I_{X_1} = (x_1, \dots, x_m, x_{m+1} - x_0, x_{m+3}, \dots, x_n) \cap (x_{m+1}, \dots, x_n).$$

Consider the flat family  $\{X_\lambda\}_{\lambda \in k}$ , where for any fixed  $\lambda \in k$ ,  $X_\lambda$  is the union of  $\Pi$  and the line

$$x_1 = \dots = x_m = x_{m+1} - \lambda x_0 = x_{m+3} = \dots = x_n = 0.$$

The ideal of  $X_\lambda$  is

$$\begin{aligned} I_{X_\lambda} &= (x_1, \dots, x_m, x_{m+1} - \lambda x_0, x_{m+3}, \dots, x_n) \cap (x_{m+1}, \dots, x_n) \\ &= (x_1, \dots, x_m, x_{m+1} - \lambda x_0) \cap (x_{m+1}, x_{m+2}) + (x_{m+3}, \dots, x_n) \\ &= (x_1, \dots, x_m, x_{m+1} - \lambda x_0) \cdot (x_{m+1}, x_{m+2}) + (x_{m+3}, \dots, x_n) \\ &= (x_1 x_{m+1}, \dots, x_m x_{m+1}, (x_{m+1} - \lambda x_0) x_{m+1}) \\ &\quad + (x_1 x_{m+2}, \dots, x_m x_{m+2}, (x_{m+1} - \lambda x_0) x_{m+2}) \\ &\quad + (x_{m+3}, \dots, x_n), \end{aligned}$$

which for  $\lambda = 0$  gives:

$$\begin{aligned} I_{X_0} &= (x_1 x_{m+1}, \dots, x_m x_{m+1}, x_{m+1}^2) + (x_1 x_{m+2}, \dots, x_m x_{m+2}, x_{m+1} x_{m+2}) \\ &\quad + (x_{m+3}, \dots, x_n), \\ &= (x_1, \dots, x_{m+1}) \cdot (x_{m+1}, x_{m+2}) + (x_{m+3}, \dots, x_n). \end{aligned}$$

Let  $(x_{m+3}, \dots, x_n) = J$ . We will prove that

$$\begin{aligned} (1) \quad I_{X_0} &= (x_1, \dots, x_{m+1}) \cdot (x_{m+1}, x_{m+2}) + J \\ &= [(x_1, \dots, x_{m+1}) + J] \cap [(x_{m+1}, x_{m+2}) + J] \cap [(x_1, \dots, x_{m+2})^2 + J]. \end{aligned}$$

We use Dedekind's Modular Law several times in what follows (see [AM69, page 6]). We start by considering the intersection of the first two ideals, i.e.,

$$\begin{aligned} & [(x_1, \dots, x_{m+1}) + J] \cap [(x_{m+1}, x_{m+2}) + J] \\ &= [(x_1, \dots, x_{m+1}) + J] \cap [((x_{m+1}) + J) + (x_{m+2})] \\ &= ((x_{m+1}) + J) + \{[(x_1, \dots, x_{m+1}) + J] \cap (x_{m+2})\} \\ &= ((x_{m+1}, x_1 x_{m+2}, \dots, x_m x_{m+2}) + J). \end{aligned}$$

It remains to intersect this last ideal with the third ideal above, i.e.,

$$\begin{aligned} & ((x_{m+1}, x_1 x_{m+2}, \dots, x_m x_{m+2}) + J) \cap [(x_1, \dots, x_{m+2})^2 + J] \\ &= [(x_{m+1}) + ((x_1 x_{m+2}, \dots, x_m x_{m+2}) + J)] \cap [(x_1, \dots, x_{m+2})^2 + J] \\ &= [(x_{m+1}) \cap ((x_1, \dots, x_{m+2})^2 + J)] + ((x_1 x_{m+2}, \dots, x_m x_{m+2}) + J) \\ &= \{(x_{m+1}) \cap [(x_{m+1}) \cdot (x_1, \dots, x_{m+2}) + (x_1, \dots, x_m, x_{m+2})^2 + J]\} \\ &\quad + ((x_1 x_{m+2}, \dots, x_m x_{m+2}) + J) \\ &= [(x_{m+1}) \cdot (x_1, \dots, x_{m+2})] + [(x_{m+1}) \cap ((x_1, \dots, x_m, x_{m+2})^2 + J)] \\ &\quad + ((x_1 x_{m+2}, \dots, x_m x_{m+2}) + J). \end{aligned}$$

Clearly the middle ideal is contained in the sum of the other two, and so the last ideal is equal to

$$\begin{aligned} & [(x_{m+1}) \cdot (x_1, \dots, x_{m+2})] + ((x_1 x_{m+2}, \dots, x_m x_{m+2}) + J) \\ &= (x_1, \dots, x_{m+1}) \cdot (x_{m+1}, x_{m+2}) + J. \end{aligned}$$

So we have proved that  $I_{X_0}$  is

$$[(x_1, \dots, x_{m+1}) + J] \cap [(x_{m+1}, x_{m+2}) + J] \cap [(x_1, \dots, x_{m+2})^2 + J].$$

Since  $J$  is the ideal of  $\langle X_1 \rangle$ , the first ideal in this intersection defines a line  $L$  in  $\langle X_1 \rangle$  which meets the linear space  $\Pi$  (defined by the second ideal in this intersection) in the point  $P = [1 : 0 : \dots : 0] \in \mathbb{P}^n$ , which is the support of the third ideal in this intersection. The third ideal, in fact, describes the scheme  $2P|_{\langle X_1 \rangle}$  which is the double point  $2P$  of  $\mathbb{P}^n$  restricted to the span of  $X_1$ .

The ideal of  $H$  is  $(x_{m+1}) + J$ , hence from (1) we have that the ideal of  $\text{Res}_H(X_0)$  is

$$\begin{aligned} I_{X_0} : I_H &= [(x_1, \dots, x_{m+1}) \cdot (x_{m+1}, x_{m+2}) + J] : ((x_{m+1}) + J) \\ &= (x_1, \dots, x_n) = I_P. \end{aligned}$$

□

**Definition 2.6.** We say that  $C$  is a *degenerate conic* if  $C$  is the union of two intersecting lines  $L_1, L_2$ . In this case we write  $C = L_1 + L_2$ .

**Definition 2.7.** Let  $n \geq m + 2$ . Let  $\Pi \simeq \mathbb{P}^m \subset \mathbb{P}^n$  be a linear space of dimension  $m$ , let  $P \in \Pi$  be a point and let  $L \not\subset \Pi$  be a generic line through  $P$ . Let  $T \simeq \mathbb{P}^{m+2}$  be a generic linear space containing the scheme  $L + \Pi$ . We call the scheme  $L + \Pi + 2P|_T$  an  $(m+2)$ -dimensional sundial. (See, for instance, the scheme  $X_0$  of Lemma 2.5).

Note that for  $m = 1$ , the scheme  $L + \Pi$  is a degenerate conic and the 3-dimensional sundial  $L + \Pi + 2P|_T$  is a *degenerate conic with an embedded point* (see [HH82]).

**Theorem 2.8** (Hartshorne-Hirschowitz, [HH82]). *Let  $n, d \in \mathbb{N}$ . For  $n \geq 3$ , the ideal of the scheme  $X \subset \mathbb{P}^n$  consisting of  $s$  generic lines has the expected dimension, that is,*

$$\dim(I_X)_d = \max \left\{ \binom{d+n}{n} - s(d+1), 0 \right\},$$

or equivalently

$$H(X, d) = \min \left\{ hp(\mathbb{P}^n, d) = \binom{d+n}{n}, hp(X, d) = s(d+1) \right\}.$$

□

Since a line imposes at most  $d+1$  conditions to the forms of degree  $d$ , the first part of the following lemma is clear. The second statement of the lemma is obvious.

**Lemma 2.9.** *Let  $n, d, s \in \mathbb{N}$ ,  $n \geq 4$ . Let  $\Pi \subset \mathbb{P}^n$  be a plane, and let  $L_1, \dots, L_s \subset \mathbb{P}^n$  be  $s$  generic lines. Let*

$$X_s = \Pi + L_1 + \dots + L_s \subset \mathbb{P}^n.$$

- (i) *If  $\dim(I_{X_s})_d = \binom{d+n}{n} - \binom{d+2}{2} - s(d+1)$ , then  $\dim(I_{X_{s'}})_d = \binom{d+n}{n} - \binom{d+2}{2} - s'(d+1)$  for any  $s' < s$ .*
- (ii) *If  $\dim(I_{X_s})_d = 0$ , then  $\dim(I_{X_{s'}})_d = 0$  for any  $s' > s$ .*

□

### 3. THE BASE FOR OUR INDUCTION

In this section we prove our main Theorem (see 5.1) in  $\mathbb{P}^4$ .

**Theorem 3.1.** *Let  $d \in \mathbb{N}$  and  $\Pi \subset \mathbb{P}^4$  be a plane, and let  $L_1, \dots, L_s \subset \mathbb{P}^4$  be  $s$  generic lines. Set*

$$X = \Pi + L_1 + \dots + L_s \subset \mathbb{P}^4.$$

*Then*

$$\dim(I_X)_d = \max \left\{ \binom{d+4}{4} - \binom{d+2}{2} - s(d+1), 0 \right\},$$

or equivalently  $X$  has bipolynomial Hilbert function.

*Proof.* We proceed by induction on  $d$ . Since the theorem is obvious for  $d = 1$ , let  $d > 1$ . By Lemma 2.9 it suffices to prove the theorem for  $s = e$  and  $s = e^*$ , where

$$e = \left\lfloor \frac{\binom{d+4}{4} - \binom{d+2}{2}}{d+1} \right\rfloor = \left\lfloor \frac{d(d+2)(d+7)}{24} \right\rfloor; \quad e^* = \left\lceil \frac{\binom{d+4}{4} - \binom{d+2}{2}}{d+1} \right\rceil.$$

Let

$$\bar{e} = \left\lfloor \frac{\binom{(d-1)+4}{4} - \binom{(d-1)+2}{2}}{(d-1)+1} \right\rfloor = \left\lfloor \frac{(d-1)(d+1)(d+6)}{24} \right\rfloor.$$

We consider two cases.

*Case 1:  $d$  odd.*

For  $s = e$ , we have to prove that  $\dim(I_X)_d = \binom{d+4}{4} - \binom{d+2}{2} - e(d+1)$  (which is obviously positive). Since  $\dim(I_X)_d \geq \binom{d+4}{4} - \binom{d+2}{2} - e(d+1)$ , we have only to show that  $\dim(I_X)_d \leq \binom{d+4}{4} - \binom{d+2}{2} - e(d+1)$ .

For  $s = e^*$ , we have to prove that  $\dim(I_X)_d = 0$ .

In order to prove these statements we construct a scheme  $Y$  obtained from  $X$  by specializing the  $s - \bar{e}$  lines  $L_{\bar{e}+1}, \dots, L_s$  into a generic hyperplane  $H \simeq \mathbb{P}^3$  (we can do this since  $\bar{e} < s$ ).

If we can prove that  $\dim(I_Y)_d = \max\{\binom{d+4}{4} - \binom{d+2}{2} - s(d+1); 0\}$ , that is, if we can show that the plane and the  $s$  lines give the expected number of conditions to the forms of degree  $d$  of  $\mathbb{P}^4$ , then (by the semicontinuity of the Hilbert function) we are done.

Note that

$$\text{Res}_H Y = L_1 + \dots + L_{\bar{e}} + \Pi \subset \mathbb{P}^4,$$

and

$$\text{Tr}_H Y = P_1 + \dots + P_{\bar{e}} + L_{\bar{e}+1} + \dots + L_s + L \subset \mathbb{P}^3,$$

where  $P_i = L_i \cap H$ , ( $1 \leq i \leq \bar{e}$ ), and  $L$  is the line  $\Pi \cap H$ .

Since  $d$  is odd, the number  $\frac{(d-1)(d+1)(d+6)}{24}$  is an integer, so

$$\bar{e} = \frac{(d-1)(d+1)(d+6)}{24}.$$

The inductive hypothesis applied to  $\text{Res}_H Y$  in degree  $d-1$  yields:

$$\dim(I_{\text{Res}_H Y})_{d-1} = \binom{d+3}{4} - \binom{d+1}{2} - \bar{e}(d) = 0.$$

By Theorem 2.8, since the  $P_i$  are generic points, we get

$$\dim(I_{\text{Tr}_H Y})_d = \max \left\{ \binom{d+3}{3} - \bar{e} - (s - \bar{e} + 1)(d+1); 0 \right\}$$

$$\begin{aligned}
&= \max \left\{ \binom{d+3}{3} + \bar{e}d - (s+1)(d+1); 0 \right\} \\
&= \begin{cases} \binom{d+4}{4} - \binom{d+2}{2} - e(d+1) & \text{for } s = e \\ 0 & \text{for } s = e^* \end{cases},
\end{aligned}$$

and the conclusion follows by Lemma 2.4 with  $\delta = 1$ .

*Case 2:  $d$  even.*

In this case  $e = e^* = \frac{d(d+2)(d+7)}{24}$ , and so we only have to prove that  $\dim(I_X)_d = 0$ . Let

$$x = \frac{d(d+2)}{8},$$

and note that  $x$  is an integer,  $x < e$ .

Let  $H \simeq \mathbb{P}^3$  be a generic hyperplane containing the plane  $\Pi$ , and let  $Y$  be the scheme obtained from  $X$  by degenerating the  $x$  lines  $L_1, \dots, L_x$  into  $H$ . By abuse of notation, we will again denote these lines by  $L_1, \dots, L_x$ . By Lemma 2.5, with  $m = 2$ , we get

$$Y = L_1 + \dots + L_x + 2P_1 + \dots + 2P_x + \Pi + L_{x+1} + \dots + L_e,$$

where  $P_i = L_i \cap \Pi$  ( $1 \leq i \leq x$ ) and the  $2P_i$  are double points in  $\mathbb{P}^4$ . If we can prove that  $\dim(I_Y)_d = 0$  we are done.

By Lemma 2.5, with  $m = 2$ , we get

$$\text{Res}_H Y = P_1 + \dots + P_x + L_{x+1} + \dots + L_e \subset \mathbb{P}^4,$$

where the  $P_i$  are generic points in  $\Pi$ .

Also,

$$\text{Tr}_H Y = L_1 + \dots + L_x + 2P_1|_H + \dots + 2P_x|_H + \Pi + Q_{x+1} + \dots + Q_e,$$

but, since  $2P_i|_H \subset L_i + \Pi$ , we get

$$\text{Tr}_H Y = L_1 + \dots + L_x + \Pi + Q_{x+1} + \dots + Q_e \subset \mathbb{P}^3$$

where  $Q_i = L_i \cap H$ , ( $x+1 \leq i \leq e$ ).

Since  $\Pi$  is a fixed component of the zero locus for the forms of  $I_{Y \cap H}$ , we get that

$$\dim(I_{\text{Tr}_H Y})_d = \dim(I_{\text{Tr}_H Y - \Pi})_{d-1}.$$

Since the  $Q_i$  are generic points, we can apply Theorem 2.8 and get

$$(2) \quad \dim(I_{\text{Tr}_H Y - \Pi})_{d-1} = \binom{d-1+3}{3} - xd - (e-x) = 0.$$

Now we will prove that  $\dim(I_{\text{Res}_H Y})_{d-1} = 0$ .

By Theorem 2.8 we know that

$$(3) \quad \dim(I_{L_{x+1} + \dots + L_e})_{d-1} = \binom{d+3}{4} - d(e-x) = x.$$



Moreover, since the scheme  $\Pi + L_{x+1} + \cdots + L_e$  has  $e - x$  lines, and it is easy to show that

$$e - x = \frac{d(d+2)(d+4)}{24} \geq \left\lceil \frac{\binom{(d-1)+4}{4} - \binom{(d-1)+2}{2}}{(d-1)+1} \right\rceil = \left\lceil \frac{(d-1)(d+1)(d+6)}{24} \right\rceil,$$

then, by the inductive hypothesis, we get

$$(4) \quad \dim(I_{\Pi+L_{x+1}+\cdots+L_e})_{d-1} = 0.$$

Now we apply Lemma 2.2; by (3) and (4) we have

$$(5) \quad \dim(I_{Res_H Y})_{d-1} = 0.$$

Finally, by (2), (5) and Lemma 2.4 (with  $\delta = 1$ ) we get  $\dim(I_Y)_d = 0$ , and that completes the proof of our main theorem for  $\mathbb{P}^4$ .  $\square$

#### 4. SOME TECHNICAL LEMMATA

Although the base case for an inductive approach to our main theorem was relatively straightforward, this is not the case for the inductive step.

One aspect is relatively clear. We first specialize some lines and degenerate other pairs of lines and divide our calculation, via Castelnuovo, into a Residual scheme (which we can handle easily) and a Trace scheme in a lower dimensional projective space. It is here that the difficulties take place. The Trace scheme will consist of degenerate conics, points and lines. Unfortunately, it is not always the case that generic collections of degenerate conics behave well with respect to postulational questions. The following example makes that clear.

*Remark 4.1.* If  $C$  is a degenerate conic in  $\mathbb{P}^3$  then imposing the passage through  $C$  imposes 7 conditions on the cubics of  $\mathbb{P}^3$ . One might then suspect that if  $X$  is the union of three generic degenerate conics in  $\mathbb{P}^3$  then  $X$  would impose  $3 \cdot 7 = 21$  conditions on cubics. I.e. there would not be a cubic surface through  $X$ , although there obviously is one.

It is the existence of such examples that complicates the induction step. In fact, to get around this difficulty, we have to consider (at the same time) several auxiliary families combining both specializations and degenerations of a scheme consisting of a collection of generic lines and points.

Note that the first two lemmata deal with such families of auxiliary schemes in  $\mathbb{P}^3$ . These are needed to deal with the Trace scheme in  $\mathbb{P}^4$  which occurs in the first inductive step from  $\mathbb{P}^4$  to  $\mathbb{P}^5$ . These two lemmata also serve to point out the kinds of families we will need for the remainder of the proof.

**Lemma 4.2.** *Let  $d = 2(4h + r + 1)$ ,  $h \in \mathbb{N}$ ,  $r = 0; 1; 3$ , (that is,  $d \equiv 0; 2; 4, \text{ mod } 8$ ). Let*

$$c = \left\lfloor \frac{\binom{d+3}{4}}{d} \right\rfloor,$$

and set

$$a = \binom{d+3}{4} - dc; \quad b = \frac{\binom{d+3}{3} - a(2d+1) - c}{d+1}.$$

Then

- (i)  $b$  is an integer;
- (ii) if  $x = \binom{d+1}{3} - (a+b)(d-1)$  we have  $0 \leq x < c$ ;
- (iii) if  $W \subset \mathbb{P}^3$  is the following scheme

$$W = C_1 + \cdots + C_a + M_1 + \cdots + M_b + P_1 + \cdots + P_c$$

(where the  $C_i$  are generic degenerate conics, the  $M_i$  are generic lines, and the  $P_i$  are generic points) then  $W$  gives the expected number of conditions to the forms of degree  $d$ , that is

$$\dim(I_W)_d = \binom{d+3}{3} - a(2d+1) - b(d+1) - c = 0.$$

*Proof.* (i) An easy computation, yields

- for  $d = 8h + 2$  (that is for  $r = 0$ ),

$$c = \frac{1}{4} \binom{d+3}{3} - \frac{1}{2}; \quad a = \frac{d}{2} = 4h + 1; \quad \text{and so } b = 8h^2 + h + 1;$$

- for  $d = 8h + 4$  (that is for  $r = 1$ ),

$$c = \frac{1}{4} \binom{d+3}{3} - \frac{3}{4}; \quad a = \frac{3d}{4} = 6h + 3; \quad \text{and so } b = 8h^2 + h;$$

- for  $d = 8h + 8$  (that is for  $r = 3$ ),

$$c = \frac{1}{4} \binom{d+3}{3} - \frac{1}{4}; \quad a = \frac{d}{4} = 2h + 1; \quad \text{and so } b = 8h^2 + 17h + 10.$$

(ii) Using (i) and direct computation, (ii) easily follows.

(iii) Observe that

$$\begin{aligned} & \binom{d+3}{3} - a(2d+1) - b(d+1) - c \\ &= \binom{d+3}{3} - a(2d+1) - \left( \binom{d+3}{3} + a(2d+1) + c - c \right) = 0. \end{aligned}$$

Thus we have to prove that  $\dim(I_W)_d = 0$ .

If  $d = 2$ , that is, for  $h = r = 0$ , we have  $a = 1$ ,  $b = 1$ ,  $c = 2$ , and it is easy to see that there are not quadrics containing the scheme  $C_1 + M_1 + P_1 + P_2$ .

Let  $d > 2$ . Let  $L_{i,1}, L_{i,2}$  be the two lines which form the degenerate conic  $C_i$ , and let  $Q$  be a smooth quadric surface. Let  $x$  be as in (ii) and let  $\widetilde{W}$  be the scheme obtained from  $W$  by specializing  $(c - x)$  of the  $c$  simple points  $P_i$  to generic points on  $Q$  and by specializing the conics  $C_i$  in such a way that the lines  $L_{1,1}, \dots, L_{a,1}$  become lines of the same ruling on  $Q$ , (the lines  $L_{1,2}, \dots, L_{a,2}$  remain generic lines, not lying on  $Q$ ).

$L_{i,2}$  meets  $Q$  in the two points which are  $(L_{i,1} \cap L_{i,2})$  and another, which we denote by  $R_{i,2}$ . In the same way,  $M_i$  meets  $Q$  in the two points  $S_{i,1}, S_{i,2}$ . We have

$$\text{Res}_Q \widetilde{W} = L_{1,2} + \dots + L_{a,2} + M_1 + \dots + M_b + P_1 + \dots + P_x \subset \mathbb{P}^4,$$

where the  $L_{i,2}$  and the  $M_i$  are generic lines. By Theorem 2.8 and the description of  $x$  we get

$$\dim(I_{\text{Res}_Q \widetilde{W}})_{d-2} = \binom{d+1}{3} - (a+b)(d-1) - x = 0.$$

Now consider  $\text{Tr}_Q \widetilde{W}$ , which is

$$L_{1,1} + \dots + L_{a,1} + R_{1,2} + \dots + R_{a,2} + S_{1,1} + S_{1,2} + \dots + S_{b,1} + S_{b,2} + P_{x+1} + \dots + P_c.$$

Note that the points  $R_{i,2}, (1 \leq i \leq a); S_{i,1}, S_{i,2}, (1 \leq i \leq b); P_i, (x+1 \leq i \leq c)$  are generic points on  $Q$  and the lines all come from the same ruling on  $Q$ , hence

$$\dim(I_{\text{Tr}_Q \widetilde{W}})_d = (d-a+1)(d+1) - a - 2b - (c-x).$$

By a direct computation, we get  $\dim(I_{\text{Tr}_Q \widetilde{W}})_d = 0$ .

So by Lemma 2.4, with  $n = 3$  and  $\delta = 2$ , the conclusion follows.  $\square$

**Lemma 4.3.** *Let  $d \geq 3$  be odd, or  $d = 8h + 6$ ,  $h \in \mathbb{N}$ , (that is,  $d \equiv 1; 3; 5; 6; 7, \text{ mod } 8$ ). Let*

$$c = \frac{\binom{d+3}{4}}{d}$$

and set

$$b = \left\lfloor \frac{\binom{d+4}{4}}{d+1} \right\rfloor - c - 2; \quad b^* = \left\lceil \frac{\binom{d+4}{4}}{d+1} \right\rceil - c - 2.$$

Then

- (i)  $b > 0$  and  $c$  is an integer;

- (ii) if  $x = \binom{d+1}{3} - b(d-1)$ , then  $0 \leq x < c$ ;
- (iii) if  $W, W^* \subset \mathbb{P}^3$  are the following schemes

$$W = C + 2P + M_1 + \cdots + M_b + P_1 + \cdots + P_c,$$

$$W^* = C + 2P + M_1 + \cdots + M_{b^*} + P_1 + \cdots + P_c,$$

(where  $C = L_1 + L_2$  is a degenerate conic, formed by the two lines  $L_1, L_2$ ; where  $2P$  is a double point with support in  $P = L_1 \cap L_2$ ; where the  $M_i$  are generic lines and the  $P_i$  are generic points) then  $W$  and  $W^*$  give the expected number of conditions to the forms of degree  $d$ , that is

$$\dim(I_W)_d = \binom{d+3}{3} - (2d+2) - b(d+1) - c,$$

$$\text{and } \dim(I_{W^*})_d = 0.$$

*Proof.* Computing directly it is easy to verify (i) and (ii).

(iii) Since the scheme  $C + 2P$  is a degeneration of two skew lines it imposes  $2d+2$  conditions to forms of degree  $d$  (see Lemma 2.5). It follows that

$$\dim(I_W)_d \geq \binom{d+3}{3} - (2d+2) - b(d+1) - c.$$

Hence, it suffices to prove that  $\dim(I_W)_d \leq \binom{d+3}{3} - (2d+2) - b(d+1) - c$ .

Let  $Q$  be a smooth quadric surface. Let  $x$  be defined as in (ii) and let  $\widetilde{W}$  be the scheme obtained from  $W$  by specializing  $(c-x)$  of the  $c$  simple points  $P_i$  onto  $Q$  and by specializing the line  $M_1$  and the conic  $C$  in such a way that the lines  $M_1$  and  $L_1$  become lines of the same ruling on  $Q$  (the line  $L_2$  remain a generic line, not lying on  $Q$ , while the point  $P$  becomes a point lying on  $Q$ ). We have  $L_2 \cap Q = P + R$ , and set  $M_i \cap Q = S_{i,1} + S_{i,2}$ , ( $2 \leq i \leq b$ ). Then

$$\text{Res}_Q \widetilde{W} = L_2 + M_2 + \cdots + M_b + P_1 + \cdots + P_x \subset \mathbb{P}^4;$$

$$\text{Tr}_Q \widetilde{W} = L_1 + M_1 + 2P|_Q + R + S_{2,1} + S_{2,2} + \cdots + S_{b,1} + S_{b,2} + P_{x+1} + \cdots + P_c.$$

By Theorem 2.8 we immediately get

$$\dim(I_{\text{Res}_Q \widetilde{W}})_{d-2} = \binom{d+1}{3} - b(d-1) - x = 0.$$

Thinking of  $Q$  as  $\mathbb{P}^1 \times \mathbb{P}^1$ , we see that the forms of degree  $d$  in the ideal of  $L_1 + M_1 + 2P|_Q$  are curves of type  $(d-2, d)$  in  $\mathbb{P}^1 \times \mathbb{P}^1$  passing

through  $P$ , since  $P$  already belongs to  $L_1$ . With that observation, it is easy to check that

$$\begin{aligned} \dim(I_{Tr_Q \widetilde{W}})_d &= (d-1)(d+1) - 2 - 2(b-1) - c + x \\ &= \binom{d+3}{3} - (2d+2) - b(d+1) - c. \end{aligned}$$

So by Lemma 2.4, with  $n = 3$  and  $\delta = 2$ , it follows that

$$\dim(I_W)_d = \binom{d+3}{3} - (2d+2) - b(d+1) - c,$$

and we are finished with the schemes  $W$ .

We now consider the schemes  $W^*$ . If  $b = b^*$  (i.e., if  $d \equiv 5, 6, \text{ mod } 8$ ), we have  $W^* = W$ . In this case it is easy to verify that the number

$$\binom{d+3}{3} - (2d+2) - b(d+1) - c$$

is zero and so we are done.

So we are left with the case  $b^* = b + 1$ . Let  $\widetilde{W}^*$  be the scheme obtained from  $W^*$  by specializing  $(c - x)$  of the  $c$  simple points  $P_i$ , the lines  $M_1$  and  $M_2$  and the conic  $C$  in such a way that the lines  $M_1, M_2, L_1$  are lines of the same ruling on  $Q$ , and the line  $L_2$  remains a generic line not lying on  $Q$ . Note that the point  $P$  becomes a point of  $Q$ .

Set  $L_2 \cap Q = P + R$ , and set  $M_i \cap Q = S_{i,1} + S_{i,2}$ , ( $3 \leq i \leq b^*$ ). We have

$$Res_Q \widetilde{W} = L_2 + M_3 + \cdots + M_{b^*} + P_1 + \cdots + P_x \subset \mathbb{P}^3.$$

and

$$\begin{aligned} Tr_Q \widetilde{W} &= L_1 + M_1 + M_2 + 2P|_Q + R \\ &+ S_{3,1} + S_{3,2} + \cdots + S_{b^*,1} + S_{b^*,2} + P_{x+1} + \cdots + P_c. \end{aligned}$$

By Theorem 2.8 we immediately get

$$\dim(I_{Res_Q \widetilde{W}})_{d-2} = \binom{d+1}{3} - b(d-1) - x = 0.$$

Using the same reasoning as above, it is easy to check that

$$\dim(I_{Tr_Q \widetilde{W}})_d = \max \{0; (d-2)(d+1) - 2 - 2(b-1) - c + x\} = 0.$$

By Lemma 2.4, with  $n = 3$  and  $\delta = 2$ , it follows that  $\dim(I_{W^*})_d = 0$ .  $\square$

We now formalize what we did in these last lemmata.

Let  $n, d, a, b, c, \in \mathbb{N}$ ,  $n \geq 3$ ,  $d > 0$ ,  $a + b \leq d - 1$ , and let

$$t = \left\lfloor \frac{\binom{d+n}{n}}{d+1} \right\rfloor; \quad t^* = \left\lceil \frac{\binom{d+n}{n}}{d+1} \right\rceil.$$

Let  $c \leq t - 2(a + b)$ ,  $c^* \geq t^* - 2(a + b)$ . Let  $\widehat{C}_i$  be a 3-dimensional sundial (see Definition 2.7), that is a generic degenerate conic with an embedded point, and let  $M_i$  be a generic line.

Note that  $t \geq 2(d - 1)$ .

Consider the following statements:

- $S(n, d)$ : *The scheme*  
 $W(n, d) = \widehat{C}_1 + \cdots + \widehat{C}_{d-1} + M_1 + \cdots + M_{t-2(d-1)} \subset \mathbb{P}^n$ ,  
*imposes the expected number of conditions to forms of degree  $d$ , that is:*  
 $\dim(I_{W(n,d)})_d = \binom{d+n}{n} - (2d+2)(d-1) - (d+1)(t-2(d-1))$   
 $= \binom{d+n}{n} - t(d+1);$
- $S^*(n, d)$ : *The scheme*  
 $W^*(n, d) = \widehat{C}_1 + \cdots + \widehat{C}_{d-1} + M_1 + \cdots + M_{t^*-2(d-1)} \subset \mathbb{P}^n$ ,  
*imposes the expected number of conditions to forms of degree  $d$ , that is:*  
 $\dim(I_{W^*(n,d)})_d = 0.$
- $S(n, d; a, b, c)$ : *The scheme*  
 $W(n, d; a, b, c) = \widehat{C}_1 + \cdots + \widehat{C}_a + D_1 + \cdots + D_b + R_1 + \cdots + R_b + M_1 + \cdots + M_c \subset \mathbb{P}^n$ ,  
*where the  $D_i$  are generic degenerate conics, and the  $R_i$  are generic points, imposes the expected number of conditions to forms of degree  $d$ , that is:*  
 $\dim(I_{W(n,d;a,b,c)})_d = \binom{d+n}{n} - (2a+2b+c)(d+1).$
- $S^*(n, d; a, b, c^*)$ : *The scheme*  
 $W^*(n, d; a, b, c^*) = \widehat{C}_1 + \cdots + \widehat{C}_a + D_1 + \cdots + D_b + R_1 + \cdots + R_b + M_1 + \cdots + M_{c^*} \subset \mathbb{P}^n$ ,  
*where the  $D_i$  are generic degenerate conics, and the  $R_i$  are generic points, imposes the expected number of conditions to forms of degree  $d$ , that is:*  
 $\dim(I_{W(n,d;a,b,c^*)})_d = 0.$

**Lemma 4.4.** *Notation as above,*

- (i) *if  $S(n, d)$  holds, then  $S(n, d; a, b, c)$  holds;*
- (ii) *if  $S^*(n, d)$  holds, then  $S^*(n, d; a, b, c^*)$  holds.*

*Proof.* A degenerate conic with an embedded point is either a degeneration of two generic lines, or a specialization of a scheme which is the union of a degenerate conic and a simple generic point. Then by the semicontinuity of the Hilbert function, and since a line imposes at most  $d + 1$  conditions to the forms of degree  $d$ , we get (i).

(ii) immediately follows from the semicontinuity of the Hilbert function.  $\square$

**Lemma 4.5.** *Notation as above, let*

$$t = \left\lfloor \frac{\binom{d+4}{4}}{d+1} \right\rfloor; \quad t^* = \left\lceil \frac{\binom{d+4}{4}}{d+1} \right\rceil.$$

*Then  $S(4, d)$  and  $S^*(4, d)$  hold, that is*

$$\dim(I_{W(4,d)})_d = \binom{d+4}{4} - t(d+1) \quad \text{and} \quad \dim(I_{W^*(4,d)})_d = 0,$$

*where*

$$W(4, d) = \widehat{C}_1 + \cdots + \widehat{C}_{d-1} + M_1 + \cdots + M_{t-2(d-1)} \subset \mathbb{P}^4,$$

$$W^*(4, d) = \widehat{C}_1 + \cdots + \widehat{C}_{d-1} + M_1 + \cdots + M_{t^*-2(d-1)} \subset \mathbb{P}^4,$$

$$\text{and } \widehat{C}_i = C_i + 2P_i|_{H_i} = L_{i,1} + L_{i,2} + 2P_i|_{H_i}.$$

*Proof.* By induction on  $d$ . For  $d = 1$  both conclusions follows from Theorem 2.8.

Let  $d > 1$ .

We consider two cases:

*Case 1:*  $d = 2(4h + r + 1)$ ,  $h \in \mathbb{N}$ ,  $r = 0; 1; 3$ , (that is,  $d \equiv 0; 2; 4$ , mod 8). In this case  $t = t^*$ , and we will prove that  $\dim(I_{W(4,d)})_d = 0$ . Consider

$$c = \left\lfloor \frac{\binom{d+3}{4}}{d} \right\rfloor \quad \text{and} \quad a = \binom{d+3}{4} - dc.$$

Note that:

- for  $d = 8h + 2$  (that is for  $r = 0$ ):

$$c = \frac{1}{4} \binom{d+3}{3} - \frac{1}{2}; \quad a = \frac{d}{2};$$

- for  $d = 8h + 4$  (that is for  $r = 1$ ):

$$c = \frac{1}{4} \binom{d+3}{3} - \frac{3}{4}; \quad a = \frac{3d}{4};$$

- for  $d = 8h + 8$  (that is for  $r = 3$ ):

$$c = \frac{1}{4} \binom{d+3}{3} - \frac{1}{4}; \quad a = \frac{d}{4}.$$

It is easy to check that

$$1 \leq a \leq d-1; \quad 0 \leq t-2a-c \leq t-2(d-1).$$

Let  $H \simeq \mathbb{P}^3$  be a generic hyperplane. Let  $W_s(4, d)$  be the scheme obtained from  $W(4, d)$  by specializing  $t-2a-c$  lines  $M_1, \dots, M_{t-2a-c}$  into  $H$  and by specializing  $a$  degenerate conics  $\widehat{C}_1, \dots, \widehat{C}_a$ , in such a way that  $L_{i,1} + L_{i,2} \subset H$ , but  $2P_i|_{H_i} \not\subset H$ , for  $1 \leq i \leq a$ .

So

$$\begin{aligned} \text{Res}_H W_s(4, d) &= P_1 + \dots + P_a + \widehat{C}_{a+1} + \dots + \widehat{C}_{d-1} \\ &\quad + M_{t-2a-c+1} + \dots + M_{t-2(d-1)} \subset \mathbb{P}^4, \end{aligned}$$

where  $P_1, \dots, P_a$  are generic points lying on  $H$ ;

$$\begin{aligned} \text{Tr}_H W_s(4, d) &= C_1 + \dots + C_a + R_{a+1,1} + R_{a+1,2} + \dots + R_{d-1,1} + R_{d-1,2} + \\ &\quad + M_1 + \dots + M_{t-2a-c} + S_{t-2a-c+1} + \dots + S_{t-2(d-1)} \subset \mathbb{P}^3, \end{aligned}$$

where  $R_{i,1} + R_{i,2} = \widehat{C}_i \cap H = L_{i,1} \cap H + L_{i,2} \cap H$  and  $S_i = M_i \cap H$ .

By Lemma 4.2, since the  $R_{i,j}$  and the  $S_i$  are

$$2(d-1-a) + (t-2(d-1)-t+2a+c) = c$$

generic points, and  $t-2a-c = b$  ( $b$  as in Lemma 4.2), we get

$$\dim(I_{\text{Tr}_H W_s(4,d)})_d = 0.$$

If we can prove that  $\dim(I_{\text{Res}_H W_s(4,d)})_{d-1} = 0$  then, by Lemma 2.4, with  $\delta = 1$ , we are done. If  $d = 2$ , we have  $a = 1$ ,  $c = 2$  and

$$\text{Res}_H W_s(4, d) = P_1 + M_2 + \dots + M_3 \subset \mathbb{P}^4.$$

Clearly  $\dim(I_{\text{Res}_H W_s(4,d)})_1 = 0$ .

Now let  $d > 2$  and set

$$X = \widehat{C}_{a+1} + \dots + \widehat{C}_{d-1} + M_{t-2a-c+1} + \dots + M_{t-2(d-1)} \subset \mathbb{P}^4,$$

(hence  $\text{Res}_H W_s(4, d) = X + P_1 + \dots + P_a$ ). So  $X$  is the union of  $d-1-a$  degenerate conics with an embedded point and  $2a+c-2(d-1)$  lines. The first step here is to show that  $X$  imposes the right number of conditions to the forms of degree  $d-1$ .

By the induction hypothesis we have that  $S(4, d-1)$  holds. Since  $d-1-a \leq d-3$ , and

$$X = \widehat{C}_{a+1} + \dots + \widehat{C}_{d-1} + M_{t-2a-c+1} + \dots + M_{t-2(d-1)}$$

is a

$$W(4, d-1; d-1-a, 0, 2a+c-2(d-1)),$$



it follows from Lemma 4.4 (i) that  $X$  imposes independent conditions to the forms of degree  $d - 1$ . Thus

$$\begin{aligned} \dim(I_X)_{d-1} &= \binom{d-1+4}{4} - d(2(d-1-a) + 2a + c - 2(d-1)) \\ &= \binom{d+3}{4} - dc = a. \end{aligned}$$

To finish the argument we apply Lemma 2.2. This requires us to prove that  $\dim(I_{X+H})_{d-1} = 0$ . But

$$\dim(I_{X+H})_{d-1} = \dim(I_X)_{d-2}.$$

For  $d = 2$ , we obviously have  $\dim(I_X)_{d-2} = 0$ .

For  $d > 2$ , by the inductive hypothesis  $S^*(4, d-2)$  holds. Since the parameters of  $X$  (perhaps with fewer lines) satisfy the restrictions necessary to use Lemma 4.4 (ii), we get that

$$S^*(4, d-2; d-1-a, 0, 2a+c-2(d-1))$$

holds, that is,  $\dim(I_X)_{d-2} = 0$ .

So, by Lemma 2.2, we have

$$\dim(I_{X+P_1+\dots+P_a})_{d-1} = \dim(I_{\text{Res}_H W_s(4,d)})_{d-1} = 0,$$

and we are done.

*Case 2:*  $d$  odd, or  $d = 8h + 6$ ,  $h \in \mathbb{N}$ , (that is,  $d = 1; 3; 5; 6; 7 \pmod{8}$ ). Let

$$c = \frac{\binom{d+3}{4}}{d}; \quad b = t - c - 2; \quad b^* = t^* - c - 2,$$

(note that  $c$  is an integer). It is easy to check that

$$0 < b \leq t - 2(d-1) \quad \text{and} \quad 0 < b^* \leq t^* - 2(d-1).$$

Let  $W_s(4, d)$  be the scheme obtained from  $W(4, d)$  by specializing the  $b$  lines  $M_1, \dots, M_b$  and  $\widehat{C}_{d-1}$  into a hyperplane  $H \simeq \mathbb{P}^3$ .

Let  $W_s^*(4, d)$  be the scheme obtained from  $W^*(4, d)$  by specializing into  $H$  the lines  $M_1, \dots, M_{b^*}$  and the degenerate conic with an embedded point  $\widehat{C}_{d-1}$ . We have

$$\text{Res}_H W_s(4, d) = \widehat{C}_1 + \dots + \widehat{C}_{d-2} + M_{b+1} + \dots + M_{t-2(d-1)} \subset \mathbb{P}^4,$$

$$\text{Res}_H W_s^*(4, d) = \widehat{C}_1 + \dots + \widehat{C}_{d-2} + M_{b^*+1} + \dots + M_{t^*-2(d-1)} \subset \mathbb{P}^4,$$

that is, both  $\text{Res}_H W_s(4, d)$  and  $\text{Res}_H W_s^*(4, d)$  are the union of  $d-2$  degenerate conics with an embedded point and  $c-2d+4$  lines. By the inductive hypothesis we immediately get

$$\dim(I_{\text{Res}_H W_s(4,d)})_{d-1} =$$

$$= \dim(I_{Res_H W_s^*(4,d)})_{d-1} = \binom{d+3}{4} - d(2(d-2) + c - 2d + 4) = 0.$$

Now we consider the traces:

$$\begin{aligned} Tr_H W_s(4, d) &= R_{1,1} + R_{1,2} + \cdots + R_{d-2,1} + R_{d-2,2} + \widehat{C}_{d-2} + \\ &\quad + M_1 + \cdots + M_b + S_{b+1} + \cdots + S_{t-2(d-1)} \subset \mathbb{P}^3, \\ Tr_H W_s^*(4, d) &= R_{1,1} + R_{1,2} + \cdots + R_{d-2,1} + R_{d-2,2} + \widehat{C}_{d-2} + \\ &\quad + M_1 + \cdots + M_{b^*} + S_{b^*+1} + \cdots + S_{t^*-2(d-1)} \subset \mathbb{P}^3, \end{aligned}$$

where  $R_{i,1} + R_{i,2} = \widehat{C}_i \cap H$ , and  $S_i = M_i \cap H$ .

$Tr_H W_s(4, d)$  is the union of  $2(d-2) + c + 4 - 2d = c$  simple generic points, a degenerate conic with an embedded point, and  $b$  lines. So, by Lemma 4.3 we get

$$\dim(I_{Tr_H W_s(4,d)})_d = \binom{d+3}{3} - (2d+2) - b(d+1) - c.$$

Thus, by Lemma 2.4, with  $\delta = 1$ , we have

$$\dim(I_{W_s(4,d)})_d \leq \binom{d+3}{3} - (2d+2) - b(d+1) - c = \binom{d+4}{4} - t(d+1).$$

Since  $\dim(I_{W(4,d)})_d \leq \dim(I_{W_s(4,d)})_d$  and  $\binom{d+4}{4} - t(d+1)$  is the expected dimension for  $(I_{W(4,d)})_d$ , we have  $\dim(I_{W(4,d)})_d = \binom{d+4}{4} - t(d+1)$ .

Finally,  $Tr_H W_s^*(4, d)$  is the union of  $c$  simple generic points, one degenerate conic with an embedded point, and  $b^*$  lines. So, by Lemma 4.3 we get

$$\dim(I_{Tr_H W_s^*(4,d)})_d = 0,$$

and by Lemma 2.4, with  $\delta = 1$ , the conclusion follows.  $\square$

**Lemma 4.6.** *Let  $n \geq 4$ ,  $d \geq 1$ ,*

$$t = \left\lfloor \frac{\binom{d+n}{n}}{d+1} \right\rfloor; \quad t^* = \left\lceil \frac{\binom{d+n}{n}}{d+1} \right\rceil.$$

*Then  $S(n, d)$  and  $S^*(n, d)$  hold, that is*

$$\dim(I_{W(n,d)})_d = \binom{d+n}{n} - t(d+1); \quad \dim(I_{W^*(n,d)})_d = 0,$$

*where*

$$\begin{aligned} W(n, d) &= \widehat{C}_1 + \cdots + \widehat{C}_{d-1} + M_1 + \cdots + M_{t-2(d-1)} \subset \mathbb{P}^n, \\ W^*(n, d) &= \widehat{C}_1 + \cdots + \widehat{C}_{d-1} + M_1 + \cdots + M_{t^*-2(d-1)} \subset \mathbb{P}^n, \end{aligned}$$

and  $\widehat{C}_i = C_i + 2P_i|_{H_i} = L_{i,1} + L_{i,2} + 2P_i|_{H_i}$ .

*Proof.* By induction on  $n + d$ . The case  $d = 1$  follows from Theorem 2.8. For  $n = 4$ , see Lemma 4.5.

Let  $n + d > 6$ ,  $d > 1$ ,  $n > 4$ . Let

$$a = \binom{d+n-1}{n} - d \left\lfloor \frac{\binom{d+n-1}{n}}{d} \right\rfloor \quad \text{and} \quad c = \left\lfloor \frac{\binom{d+n-1}{n}}{d} \right\rfloor.$$

Note that, by a direct computation, we have

$$0 \leq a \leq d-1 \quad \text{and} \quad a \leq c \leq t-2(d-1).$$

Let  $W_s(n, d)$  be the scheme obtained from  $W(n, d)$  by specializing, into a generic hyperplane  $H \simeq \mathbb{P}^{n-1}$ , the  $d-1-a$  degenerate conics with an embedded point  $\widehat{C}_{a+1}, \dots, \widehat{C}_{d-1}$  and the  $t-2(d-1)-c$  lines  $M_{c+1}, \dots, M_{t-2(d-1)}$ . We further specialize the  $a$  degenerate conics  $\widehat{C}_1, \dots, \widehat{C}_a$ , in such a way that  $L_{i,1} + L_{i,2} \subset H$ , but  $2P_i|_{H_i} \not\subset H$ , for  $1 \leq i \leq a$ .

Analogously, let  $W_s^*(n, d)$  be the scheme obtained from  $W^*(n, d)$  by specializing, into a generic hyperplane  $H \simeq \mathbb{P}^{n-1}$ , the degenerate conics with an embedded point  $\widehat{C}_{a+1}, \dots, \widehat{C}_{d-1}$ , and the  $t^*-2(d-1)-c$  lines  $M_{c+1}, \dots, M_{t^*-2(d-1)}$ . We further specialize the  $a$  degenerate conics  $\widehat{C}_1, \dots, \widehat{C}_a$ , in such a way that  $L_{i,1} + L_{i,2} \subset H$ , but  $2P_i|_{H_i} \not\subset H$ .

From these specializations we have

$$\text{Res}_H W_s(n, d) = \text{Res}_H W_s^*(n, d) = P_1 + \dots + P_a + M_1 + \dots + M_c \subset \mathbb{P}^n,$$

where  $P_1, \dots, P_a$  are generic points of  $H$ ;

$$\text{Tr}_H W_s(n, d) =$$

$$C_1 + \dots + C_a + \widehat{C}_{a+1} + \dots + \widehat{C}_{d-1} + S_1 + \dots + S_c + M_{c+1} + \dots + M_{t-2(d-1)} \subset \mathbb{P}^{n-1},$$

and

$$\text{Tr}_H W_s^*(n, d) =$$

$$C_1 + \dots + C_a + \widehat{C}_{a+1} + \dots + \widehat{C}_{d-1} + S_1 + \dots + S_c + M_{c+1} + \dots + M_{t^*-2(d-1)} \subset \mathbb{P}^{n-1},$$

where  $S_i = M_i \cap H$ .

Consider the schemes

$$X = \text{Tr}_H W_s(n, d) - (S_{a+1} + \dots + S_c)$$

$$= \widehat{C}_{a+1} + \dots + \widehat{C}_{d-1} + C_1 + \dots + C_a + S_1 + \dots + S_a + M_{c+1} + \dots + M_{t-2(d-1)} \subset \mathbb{P}^{n-1},$$

and

$$X^* = \text{Tr}_H W_s^*(n, d) - (S_{a+1} + \dots + S_c)$$

$$= \widehat{C}_{a+1} + \dots + \widehat{C}_{d-1} + C_1 + \dots + C_a + S_1 + \dots + S_a + M_{c+1} + \dots + M_{t^*-2(d-1)} \subset \mathbb{P}^{n-1}.$$

By the inductive hypothesis,  $S(n-1, d)$  holds. By a direct computation we check that

$$t - c \leq t^* - c \leq \left\lfloor \frac{\binom{d+n-1}{d}}{d+1} \right\rfloor.$$

Hence, by Lemma 4.4, we have that  $S(n-1, d; d-1-a, a, t-2(d-1)-c)$  and  $S(n-1, d; d-1-a, a, t^*-2(d-1)-c)$  hold.

It follows that  $\dim(I_X)_d$  and  $\dim(I_{X^*})_d$  are as expected, that is,

$$\begin{aligned} \dim(I_X)_d &= \binom{d+n-1}{n-1} - (d+1)(2(d-1) + t - 2(d-1) - c) \\ &= \binom{d+n-1}{n-1} - (d+1)(t-c) = \binom{d+n}{n} - t(d+1) + c - a, \end{aligned}$$

and

$$\begin{aligned} \dim(I_{X^*})_d &= \binom{d+n-1}{n-1} - (d+1)(2(d-1) + t^* - 2(d-1) - c) \\ &= \binom{d+n-1}{n-1} - (d+1)(t^*-c) = \binom{d+n}{n} - t^*(d+1) + c - a. \end{aligned}$$

Now, since  $S_{a+1}, \dots, S_c$  are generic points and  $\binom{d+n}{n} - t^*(d+1) \leq 0$ , it follows that

$$\dim(I_{Tr_H W_s(n,d)})_d = \binom{d+n}{n} - t(d+1),$$

and

$$\dim(I_{Tr_H W_s^*(n,d)})_d = \max \left\{ 0; \binom{d+n}{n} - t^*(d+1) \right\} = 0.$$

If we prove that  $\dim(I_{Res_H W_s(n,d)})_{d-1} = \dim(I_{Res_H W_s^*(n,d)})_{d-1} = 0$  the, by Lemma 2.4 with  $\delta = 1$ , we are done.

Recall that

$$Res_H W_s(n, d) = Res_H W_s^*(n, d) = P_1 + \dots + P_a + M_1 + \dots + M_c \subset \mathbb{P}^n,$$

where  $P_1, \dots, P_a$  are generic points in  $H$ . By Lemma 2.2 it suffices to prove that  $\dim(I_{M_1+\dots+M_c})_{d-1} = a$  and  $\dim(I_{M_1+\dots+M_c+H})_{d-1} = 0$ .

By Theorem 2.8 we immediately get

$$\dim(I_{M_1+\dots+M_c})_{d-1} = \binom{d+n-1}{n} - dc = a.$$

Moreover, since  $\dim(I_{M_1+\dots+M_c+H})_{d-1} = \dim(I_{M_1+\dots+M_c})_{d-2}$ , by Theorem 2.8 we have

$$\dim(I_{M_1+\dots+M_c})_{d-2} = \max \left\{ 0; \binom{d+n-2}{n} - (d-1)c \right\} = 0,$$

and the conclusion follows.  $\square$

## 5. THE GENERAL CASE

Having collected all the preliminary lemmata necessary, we are ready to prove the main theorem of the paper.

**Theorem 5.1.** *Let  $n, d \in \mathbb{N}$ ,  $n \geq 4$ ,  $d \geq 1$ . Let  $\Pi \subset \mathbb{P}^n$  be a plane, and let  $L_1, \dots, L_s \subset \mathbb{P}^n$  be  $s$  generic lines. If*

$$X = \Pi + L_1 + \dots + L_s \subset \mathbb{P}^n$$

then

$$\dim(I_X)_d = \max \left\{ \binom{d+n}{n} - \binom{d+2}{2} - s(d+1), 0 \right\},$$

or equivalently  $X$  has bipolynomial Hilbert function.

*Proof.* We proceed by induction on  $n + d$ . The result is obvious for  $d = 1$  and any  $n$ , while for  $n = 4$  see Theorem 3.1.

Let  $d > 1$ ,  $n > 4$ . By Lemma 2.9 it suffices to prove the theorem for  $s = e$  and  $s = e^*$ , where

$$e = \left\lfloor \frac{\binom{d+n}{n} - \binom{d+2}{2}}{d+1} \right\rfloor; \quad e^* = \left\lceil \frac{\binom{d+n}{n} - \binom{d+2}{2}}{d+1} \right\rceil.$$

Let

$$e_\rho = \left\lfloor \frac{\binom{d+n-1}{n} - \binom{d+1}{2}}{d} \right\rfloor; \quad \rho = \binom{d+n-1}{n} - \binom{d+1}{2} - e_\rho d;$$

$$e_T = s - e_\rho - 2\rho, \quad (s = e, e^*).$$

It is a direct computation to check that  $e - e_\rho - 2\rho \geq 0$ .

Let  $\widehat{C}_i$  be the degenerate conic with an embedded point obtained by degenerating the lines  $L_i, L_{i+1}$ ,  $1 \leq i \leq \rho$  as in Lemma 2.5 with  $m = 1$ . By abuse of notation, we write  $\widehat{C}_i$  as  $L_i + L_{i+1} + 2P_i|_{H_i}$ , (recall that  $H_i \simeq \mathbb{P}^3$  is a generic linear space through  $P_i$ ). Let  $H \simeq \mathbb{P}^{n-1}$  be a generic hyperplane. Now specialize  $\widehat{C}_1, \dots, \widehat{C}_\rho$  in such a way that  $L_i + L_{i+1} \subset H$  and  $2P_i|_{H_i} \not\subset H$ , and specialize the  $e_T$  lines  $L_{2\rho+1}, \dots, L_{2\rho+e_T}$  into  $H$  and denote by  $Y$  the resulting scheme. We have

$$\text{Res}_H Y = \Pi + P_1 + \dots + P_\rho + L_{2\rho+e_T+1} + \dots + L_s \subset \mathbb{P}^n$$

( $P_1, \dots, P_\rho$  are generic points of  $H$ ),

$$\text{Tr}_H Y = L + C_1 + \dots + C_\rho + L_{2\rho+1} + \dots + L_{2\rho+e_T} + P_{2\rho+e_T+1} + \dots + P_s \subset \mathbb{P}^{n-1}$$

where  $L = \Pi \cap H$  and  $P_i = L_i \cap H$ , for  $2\rho + e_T + 1 \leq i \leq s$ .

$Res_H Y$  is the union of one plane,  $e_\rho$  lines and  $\rho$  generic points of  $H$ . By the inductive hypothesis we have

$$\dim(I_{\Pi+L_{2\rho+e_T+1+\dots+L_s}})_{d-1} = \rho.$$

Moreover

$$\dim(I_{H+\Pi+L_{2\rho+e_T+1+\dots+L_s}})_{d-1} = \dim(I_{\Pi+L_{2\rho+e_T+1+\dots+L_s}})_{d-2} = 0,$$

(obvious, for  $d = 1, 2$ ; by induction, for  $d > 2$ ). Hence by Lemma 2.2 we get

$$\dim(I_{Res_H Y})_{d-1} = 0.$$

$Tr_H Y$  is the union of  $\rho$  degenerate conics,  $e_T + 1$  lines, and  $e_\rho$  generic points. We will compute  $\dim(I_{Tr_H Y})_d$  by using Lemma 4.4 and Lemma 4.6. We have to check that  $\rho \leq d - 1$  and  $e_\rho \leq \rho$ . The first inequality is obvious, and it is not difficult to verify the other one. So we get

$$\dim(I_{Tr_H Y})_d = \max \left\{ 0; \binom{d+n-1}{d} - (d+1)(2\rho + e_T + 1) - e_\rho \right\},$$

and from here

$$\begin{aligned} \dim(I_{Tr_H Y})_d &= \binom{d+n}{n} - \binom{d+2}{2} - s(d+1), \quad \text{for } s = e; \\ \dim(I_{Tr_H Y})_d &= 0 \quad \text{for } s = e^*. \end{aligned}$$

The conclusion now follows from Theorem 2.4, with  $\delta = 1$ . □

## 6. APPLICATIONS

We now mention two applications of Theorem 5.1. The first is to a very classical problem concerning the existence of rational normal curves having prescribed intersections with various dimensional linear subspaces of  $\mathbb{P}^n$ . For example, the classical Theorem of Castelnuovo which asserts that there exists a unique rational normal curve through  $n + 3$  generic points of  $\mathbb{P}^n$ , is the kind of result we have in mind.

The second application is to writing polynomials in several variables in a simple form. For example, the classical theorem which says that in  $S = \mathbb{C}[x_0, \dots, x_n]$  every quadratic form is a sum of at most  $n + 1$  squares of linear forms, is the kind of theorem we intend.

**Rational normal curves.** The problem of deciding whether or not there exists a rational normal curve with prescribed intersections with generic configurations of linear spaces, is well known and, in general, unsolved. Various results and applications of answers to this problem can be found in [CC07] and [CC09].

Of particular importance in such questions is the Hilbert function of the resulting configuration of linear spaces. It is for this reason that the results of this paper can be applied to such a problem.

To illustrate the relationships we will look at the following special problem (left open in [CC09]): consider in  $\mathbb{P}^4$ ,  $P_1, P_2, P_3$  generic points,  $L_1, L_2$  generic lines and  $\pi$  a generic plane. Does there exist a rational normal curve  $\mathcal{C}$  in  $\mathbb{P}^4$  such that:

- (i)  $\mathcal{C}$  passes through the  $P_i$  ( $i = 1, 2, 3$ );
- (ii)  $\deg(\mathcal{C} \cap L_i) \geq 2$  for  $i = 1, 2$ ;
- (iii)  $\deg(\mathcal{C} \cap \pi) \geq 3$ .

An expected answer is described in [CC09] and can be obtained by arguing as follows: inside the 21 dimensional parameter space for rational normal curves in  $\mathbb{P}^4$  it is expected that those satisfying the conditions enumerated above form a subvariety of codimension 20. In other words, we expect that there is a rational normal curve in  $\mathbb{P}^4$  satisfying the conditions above.

To see that this is not the case we consider the schemes

$$X = P_1 + P_2 + L_1 + L_2 + \pi,$$

$$Y = X + P_3.$$

Using Theorem 5.1 we know that  $\dim(I_X)_2 = 1$  and  $\dim(I_Y)_2 = 0$ . If  $\mathcal{C}$  existed, then  $Q \supset X$  would imply  $Q \supset \mathcal{C}$  by a standard Bezout type argument, and so we get  $Q \supset Y$ , a contradiction.

**Polynomial decompositions.** We consider the rings  $S = \mathbb{C}[x_0, \dots, x_n]$  and  $T = \mathbb{C}[y_0, \dots, y_n]$ , and we denote by  $S_d$  and  $T_d$  their homogeneous pieces of degree  $d$ . We consider  $T$  as an  $S$ -module by letting the action of  $x_i$  on  $T$  be that of partial differentiation with respect to  $y_i$ . We also use some basic notions about apolarity (for more on this see [Ger96, IK99]).

Let  $I \subset S$  be a subset and denote by  $I^\perp \subset T$  the submodule of  $T$  annihilated by every element of  $I$ . If  $I$  is an homogeneous ideal, we recall that  $(I_d)^\perp = (I^\perp)_d$ .

Given linear forms  $a, b, c, l_i, m_i \in T_1, i = 1, \dots, s$ , one can ask the following question ( $\star$ ):

*For which values of  $d$  is it true that any form  $f \in T_d$  can be written as*

$$f(y_0, \dots, y_n) = f_1(l_1, m_1) + \dots + f_s(l_s, m_s) + g(a, b, c)$$

*for suitable forms  $f_i$  and  $g$  of degree  $d$ ?*

More precisely, we ask whether the following vector space equality holds:

$$T_d = (\mathbb{C}[l_1, m_1])_d + \dots + (\mathbb{C}[l_s, m_s])_d + (\mathbb{C}[a, b, c])_d,$$

where  $(\mathbb{C}[l_i, m_i])_d$ , respectively  $(\mathbb{C}[a, b, c])_d$ , is the degree  $d$  part of the subring of  $T$  generated by the  $l_i, m_i$ 's for a fixed  $i$ , respectively generated by  $a, b$  and  $c$ . A more general question can be considered as described in [CCG09], but a complete answer is not known. We now give a complete answer in the case of  $(\star)$ .

The connection with configurations of linear spaces is given by the following results.

**Lemma 6.1.** *Let  $\Lambda \subset \mathbb{P}^n$  be an  $i$  dimensional linear space having defining ideal  $I$ . Then, for any  $d$ , we have the following:*

$$I_d^\perp = (\mathbb{C}[l_0, \dots, l_i])_d$$

where the linear forms  $l_i \in T_1$  generate  $I_1^\perp$ .

*Proof.* After a linear change of variables, we may assume

$$I = (x_0, \dots, x_{n-i-1}).$$

As this is a monomial ideal the conclusion follows by straightforward computations.  $\square$

**Proposition 6.2.** *Let  $\Lambda = \Lambda_1 + \dots + \Lambda_s \subset \mathbb{P}^n$  be a configuration of linear spaces having defining ideal  $I$  and such that  $\dim \Lambda_i = n_i$ . Then, for any  $d$ , the following holds:*

$$I_d^\perp = (\mathbb{C}[l_{1,0}, \dots, l_{1,n_1}])_d + \dots + (\mathbb{C}[l_{s,0}, \dots, l_{s,n_s}])_d$$

where the linear forms  $l_{i,j} \in T_1$  are such that the degree 1 piece of  $(l_{i,0}, \dots, l_{i,n_i})^\perp$  generates the ideal of  $\Lambda_i$ .

*Proof.* The proof follows readily from the previous lemma once we recall that  $(I \cap J)^\perp = I^\perp + J^\perp$ .  $\square$

Now we can make clear the connection with question  $(\star)$ . Given the linear forms  $a, b, c, l_i, m_i \in T_1$  for  $i = 1, \dots, s$ , we consider the ideal  $I \subset S$  generated by the degree 1 piece of  $(a, b, c)^\perp$  and the ideals  $I_i$  generated by the degree 1 pieces of  $(l_i, m_i)^\perp, i = 1, \dots, s$ . Note that  $I \cap I_1 \cap \dots \cap I_s$  is the ideal of the union of  $s$  lines and one plane in  $\mathbb{P}^n$ . Denote this scheme by  $X$ . Now we can give an answer to question  $(\star)$  using Theorem 5.1.

**Proposition 6.3.** *With notation as above, we have: the values of  $d$  answering question  $(\star)$  are exactly the ones for which  $\dim(I_X)_d = 0$ .*

$\square$



## 7. FINAL REMARKS

Theorem 5.1 gives new evidence for the conjecture we stated in the Introduction of the paper. As our conjecture deals with generic configurations of linear spaces with non-intersecting components, we would like to say something in case there are components which are forced to intersect.

Let  $\Lambda = \bigcup \Lambda_i \subset \mathbb{P}^n$  be a generic configuration of linear spaces such that  $m_i = \dim \Lambda_i \geq m_j = \dim \Lambda_j$  if  $i \geq j$ . Then, there exist components of  $\Lambda$  which intersect if and only if  $m_1 + m_2 \geq n$ .

The first interesting case where generic configurations of linear spaces have intersecting components occurs in  $\mathbb{P}^3$  by taking lines and at least one plane.

*Remark 7.1.* Theorem 5.1 is not stated in  $\mathbb{P}^3$ , but it can easily be extended to include this case. If  $X = L_1 + \dots + L_s + \Pi \subset \mathbb{P}^3$  we consider the exact sequence

$$0 \rightarrow I_{L_1+\dots+L_s}(-1) \rightarrow R \rightarrow R/I_X \rightarrow 0$$

where the first map is multiplication by a linear form defining  $\Pi$ . We can compute  $HF(X, \cdot)$  by taking dimensions in degree  $d$  and obtain:

$$HF(X, d) = \binom{d+3}{3} - \max \left\{ 0, \binom{d+2}{3} - sd \right\}$$

for  $d > 0$  and  $HF(X, 0) = 1$ . We also notice that

$$hp(X, d) = \binom{d+2}{2} + s(d+1) - s = \binom{d+3}{3} - \binom{d+2}{3} + sd.$$

Thus  $X$  has bipolynomial Hilbert function.

Hence our conjecture holds for the union of generic lines and **one** plane even in  $\mathbb{P}^3$ , where forced intersection appear. But, in general, our conjecture is false for configurations of linear spaces with intersecting components, as shown by the following example.

**Example 7.2.** Consider  $\Lambda \subset \mathbb{P}^3$  a generic configuration of linear spaces consisting of one line and three planes. By Derksen's result in [Der07] we have  $hp(\Lambda, 1) = 3$  but clearly no plane containing  $\Lambda$  exists. Hence,

$$HF(\Lambda, 1) = 4 \neq \min\{hp(\mathbb{P}^3, 4) = 4, hp(\Lambda, 4) = 3\}$$

and the Hilbert function is not bipolynomial.

We are not aware of any general result providing evidence for the behavior of  $HF(\Lambda, d)$  when the components of  $\Lambda$  are intersecting. We did, however, conduct experiments using the computer algebra system CoCoA [CoC04] and the results obtained suggest the following:

let  $\Lambda \subset \mathbb{P}^n$  be a generic configuration of linear spaces.  
There exists an integer  $d(\Lambda)$  such that

$$HF(\Lambda, d) = hp(\mathbb{P}^n, d), \text{ for } d \leq d(\Lambda)$$

and

$$HF(\Lambda, d) = hp(\Lambda, d), \text{ for } d > d(\Lambda).$$

This seems to be a reasonable possibility for the Hilbert function of generic configurations of linear spaces (even with forced intersections), but the evidence is still too sparse to call it a conjecture.

## REFERENCES

- [AH95] J. Alexander and A. Hirschowitz. Polynomial interpolation in several variables. *J. Algebraic Geom.*, 4(2):201–222, 1995.
- [AM69] M. F. Atiyah and I. G. Macdonald. *Introduction to commutative algebra*. Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1969.
- [CC07] E. Carlini and M. V. Catalisano. Existence results for rational normal curves. *J. Lond. Math. Soc. (2)*, 76(1):73–86, 2007.
- [CC09] E. Carlini and M. V. Catalisano. On rational normal curves in projective space. *J. Lond. Math. Soc. (2)*, 80(1):1–17, 2009.
- [CCG09] E. Carlini, M. V. Catalisano, and A.V. Geramita. Subspace arrangements, configurations of linear spaces and the quadrics containing them. preprint, arXiv:0909.3802v2, 2009.
- [CoC04] CoCoATeam. CoCoA: a system for doing Computations in Commutative Algebra. Available at <http://cocoa.dima.unige.it>, 2004.
- [Der07] Harm Derksen. Hilbert series of subspace arrangements. *J. Pure Appl. Algebra*, 209(1):91–98, 2007.
- [DS02] Harm Derksen and Jessica Sidman. A sharp bound for the Castelnuovo-Mumford regularity of subspace arrangements. *Adv. Math.*, 172(2):151–157, 2002.
- [Ger96] A. V. Geramita. Inverse systems of fat points: Waring’s problem, secant varieties of Veronese varieties and parameter spaces for Gorenstein ideals. In *The Curves Seminar at Queen’s, Vol. X (Kingston, ON, 1995)*, volume 102 of *Queen’s Papers in Pure and Appl. Math.*, pages 2–114. Queen’s Univ., Kingston, ON, 1996.
- [GMR83] A. V. Geramita, P. Maroscia, and L. G. Roberts. The Hilbert function of a reduced  $k$ -algebra. *J. London Math. Soc. (2)*, 28(3):443–452, 1983.
- [HH82] Robin Hartshorne and André Hirschowitz. Droites en position générale dans l’espace projectif. In *Algebraic geometry (La Rábida, 1981)*, volume 961 of *Lecture Notes in Math.*, pages 169–188. Springer, Berlin, 1982.
- [IK99] A. Iarrobino and V. Kanev. *Power sums, Gorenstein algebras, and determinantal loci*, volume 1721 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1999.

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